Hahn–Banach extension of multilinear forms and summability

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Abstract

The aim of this paper is to investigate close relations between the validity of Hahn–Banach extension theorems for multilinear forms on Banach spaces and summability properties of sequences from these spaces. A case of particular importance occurs when we consider Banach spaces which have the property that every bilinear form extends to any superspace.

Keywords: Multilinear forms; Bilinear forms; Hahn–Banach extension; Summing properties

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1. Introduction

The importance of the classical Hahn–Banach theorem in the linear theory of Banach spaces has motivated numerous attempts to establish corresponding non-linear versions. This question was treated, e.g., by S. Dineen [20] in the context of holomorphic functions on locally convex spaces; see also I. Zalduendo's recent survey article [39]. Homogeneous polynomials or, equivalently, (bounded) multilinear forms provide the easiest non-trivial examples (and building blocks) of holomorphic functions. As is immediately realized, there is no hope for a Hahn–Banach theorem to hold for general multilinear forms. But quite a number of interesting positive results about particular multilinear forms and particular spaces do exist.

At least three different lines of research can be spotted in the literature. One may fix, for example, a Banach space X and ask for those superspaces Y which have the property that every multilinear form on X extends to a multilinear form on Y. The case when Y is the bidual X^{**} of X is the topic of what is now known as the Aron–Berner extension; see, e.g., [2,16,21] or [9] for this and related results. Another line of investigation deals with the case where one starts from a fixed space X and asks for subspaces Y such that any multilinear form on Y extends to a multilinear form on X (see [14,18]). Here Maurey's extension theorem comes to mind which informs us that if Y is a subspace of a type 2 Banach space X, then every bounded bilinear form on Y extends to a bounded bilinear form on all of X; see [19,28] and [12] for a generalization. Finally, one may concentrate on multilinear forms on a Banach space X which admit an extension to a multilinear form on *any* superspace Y of X (see [9,13,26]). Such multilinear forms will prevail in our work; we call them *extendible* multilinear forms. We emphasize that preservation of norms is not required.

In the 1950s, A. Grothendieck uncovered deep connections between extendible bilinear forms and summability properties of associated operators. In particular, his fundamental theorem of the metric theory of tensor products essentially says that the extendible bilinear forms on any Banach space coincide with what we will call $(1; 2, 2)$-summing bilinear forms (definitions will be given below). We present the essence in Diagram 1.

Here we say that a Banach space X has BEP (Bilinear Extension Property) if every bilinear form on X is extendible. We also say that a pair (X, Y) of Banach spaces has BEP if every bilinear form on $X \times Y$ is extendible. The properties TEP (Trilinear Extension Property) and nEP (n-linear Extension Property) are defined analogously. Trivially, any collection of n injective Banach spaces has the nEP.

In this paper, one of the goals is to investigate how Diagram 1 changes when we pass to the trilinear (or the n-linear, $n \geqslant 3$) case. A major part of the results to be proved in subsequent sections can be summarized in Diagram 2.

We will see that for every Banach space X and for every $n \geqslant 2$ the space of extendible n-linear forms on X, $\mathcal{L}^n_{\mathrm{ext}}(X)$, is contained in the space of absolutely $(1; 2, \dots, 2)$-summing n-linear forms on X, $\mathcal{L}^n_{(1;2,\dots,2)}(X)$, which is of course part of the space $\mathcal{L}^n(X)$ of all (continuous) n-

For every Banach space X,
$\mathcal{L}^2_{\mathrm{ext}}(X) = \mathcal{L}^2_{(1;2,2)}(X) \subseteq \mathcal{L}^2(X)$,
and $\subseteq$ is $=$ if and only if X has BEP

Diagram 1.

For every Banach space X and for every $n \geqslant 3$,

$$\begin{tikzcd}
\mathcal{L}^n_{\mathrm{ext}}(X) & \underset{=}{\subset} \arrow[d, leftrightarrow] & \mathcal{L}^n_{(1;2,\dots,2)}(X) & \underset{=}{\subset} & \mathcal{L}^n(X) \\
 & X \text{ has nEP} \arrow[rr] & & X \text{ has BEP} \arrow[u] &
\end{tikzcd}$$

Diagram 2.

linear forms on X. Diagram 1 says that, in the bilinear case, the spaces $\mathcal{L}^2_{\mathrm{ext}}(X)$ and $\mathcal{L}^2_{(1;2,2)}(X)$ are the same, for every Banach space X. On the other hand, Diagram 2 shows that for $n \geqslant 3$, $\mathcal{L}^n_{\mathrm{ext}}(X) = \mathcal{L}^n_{(1;2,\dots,2)}(X)$ holds iff X has nEP. This implies that X has BEP, and we will prove that if X has BEP, then $\mathcal{L}^n_{(1;2,\dots,2)}(X) = \mathcal{L}^n(X)$ holds for every $n \geqslant 2$.

As was already noted on other occasions (see, for instance, [10,30]), several results on multilinear extension and related multilinear summing maps change dramatically when passing from the bilinear to the trilinear case. We are going to obtain further examples of this kind. Nevertheless, Diagram 2 informs us, that also in the general case, the extension problem continues to be intimately linked with summability properties of appropriate mappings.

The organization of the paper is as follows. In Section 2 we fix some notation and recall several basic definitions and facts on linear and multilinear mappings between Banach spaces, ideals of linear operators, and alike, which will repeatedly be used later on. Section 3 is devoted to the bilinear case. In particular, we analyze structural properties of Banach spaces enjoying BEP. Some of these results will be important for our discussions in particular in Section 4, where we provide the details needed to establish Diagram 2. Finally, in Section 5, we investigate to what extent our extension problem for multilinear forms can be described in terms of multilinear variants of the concept of dominated operators, as it is well known from the theory of Banach ideals.

2. Definitions and notation

We shall employ standard terminology and notation on Banach spaces and their operators. In particular, operators will be bounded linear maps between Banach spaces; also multilinear mappings and forms are always understood to be bounded. Moreover, subspaces of Banach spaces are closed linear submanifolds. Further, B_Z will be the closed unit ball of a Banach space Z. If Z happens to be the dual of some Banach space, then we will usually consider B_Z as a compact space with respect to the corresponding weak*-topology.

Given Banach spaces $X_1, \dots, X_n$ and Y,

$$\mathcal{L}^n(X_1, \dots, X_n; Y)$$

will be the standard Banach space of all n-linear mappings $X_1 \times \cdots \times X_n \to Y$. Its norm will be denoted by $\|\cdot\|$. If $X_1 = \cdots = X_n = X$, then we replace the n-tuplet $(X_1, \dots, X_n)$ simply by X and write $\mathcal{L}^n(X; Y)$ for the above space. $\mathcal{L}^1(X; Y)$ is the usual space $\mathcal{L}(X; Y)$ of operators $X \to Y$. And if Y is the basic scalar field $\mathbb{K}$, we let it just disappear from our notation:

$$\mathcal{L}^n(X_1, \dots, X_n) \quad \text{and} \quad \mathcal{L}^n(X)$$

will be the spaces of n-linear forms on $X_1 \times \cdots \times X_n$, respectively, on X^n $(= X \times \cdots \times X)$. Again, $\mathcal{L}^1(X)$ is just the usual dual X^* of X.

If X_1 and X_2 are Banach spaces, then we denote by $X_1 \tilde{\otimes}_\pi X_2$ and $X_1 \tilde{\otimes}_\varepsilon X_2$ their completed projective and injective tensor products, respectively. In a natural way, the dual of $X_1 \tilde{\otimes}_\pi X_2$ can be identified with $\mathcal{L}^2(X_1, X_2)$; in the same way, the dual of $X_1 \tilde{\otimes}_\varepsilon X_2$ is the space of Grothendieck's integral bilinear forms on $X_1 \times X_2$.

We assume familiarity with terminology, facts and constructions related to operator ideals, that is, ideals of (linear) Banach space operators in the sense of A. Pietsch. All necessary details can be found in [17,19,33]. To get started, we need to generalize some well-known concepts from this theory to multilinear mappings.

Let X be a Banach space and $1 \leqslant p < \infty$. Given a finite sequence $(x_i)_{i=1}^m$ in X, we write $\|(x_i)_{i=1}^m\|_p^w$ to denote

$$\sup\left\{\left(\sum_{i=1}^m |x^*(x_i)|^p\right)^{1/p} : x^* \in B_{X^*}\right\}.$$

A sequence $\underline{x} = (x_n)_{n\in\mathbb{N}}$ in X is said to be *weakly p-summable* if $\|\underline{x}\|_p^w = \sup_N \|(x_i)_{i=1}^N\|_p^w < \infty$. The set $\ell_p^w(X)$ of all such sequences is a Banach space, with $\underline{x} \mapsto \|\underline{x}\|_p^w$ as a norm.

The following definitions were introduced in [34] (see also [4,15,22] or [32]). Let finitely many Banach spaces $X_1, \ldots, X_n, Y$ and $T \in \mathcal{L}^n(X_1, \ldots, X_n; Y)$ be given. Let further $0 < s < \infty$ and $1 \leqslant r_1, \ldots, r_n < \infty$ be such that $\frac{1}{s} \leqslant \frac{1}{r_1} + \cdots + \frac{1}{r_n}$. We say that T is *absolutely* $(s; r_1, \ldots, r_n)$-*summing* if there exists $C \geqslant 0$ such that, however we choose finitely many vectors x_i^j from each X_j, $1 \leqslant i \leqslant m$, we have

$$\left(\sum_{i=1}^m \|T(x_i^1, \ldots, x_i^n)\|^s\right)^{1/s} \leqslant C \cdot \prod_{j=1}^n \|(x_i^j)_{i=1}^m\|_{r_j}^w.$$

The smallest C which works is denoted by $\|T\|_{(s;r_1,\ldots,r_n)}$. The set

$$\mathcal{L}^n_{(s;r_1,\ldots,r_n)}(X_1, \ldots, X_n; Y)$$

of all $(s; r_1, \ldots, r_n)$-summing n-linear maps in $\mathcal{L}^n(X_1, \ldots, X_n; Y)$ is readily seen to be a linear space. If $s \geqslant 1$, then it becomes a Banach space with $\|\cdot\|_{(s;r_1,\ldots,r_n)}$ as a norm (for $s < 1$ we get a quasi-Banach space).

In our context, a case of particular significance occurs when $\frac{1}{s} = \frac{1}{r_1} + \cdots + \frac{1}{r_n}$. Now our absolutely $(s; r_1, \ldots, r_n)$-summing multilinear mappings $T : X_1 \times \cdots \times X_n \to Y$ are called $(r_1, \ldots, r_n)$-*dominated*, and we will write

$$\mathcal{D}^n_{(r_1,\ldots,r_n)}(X_1, \ldots, X_n; Y) \quad \text{and} \quad \delta^n_{(r_1,\ldots,r_n)}(T)$$

instead of $\mathcal{L}^n_{(s;r_1,\ldots,r_n)}(X_1, \ldots, X_n; Y)$ and $\|T\|_{(s;r_1,\ldots,r_n)}$, respectively. To simplify further,

$$\mathcal{D}^n_r(X_1, \ldots, X_n; Y) \quad \text{and} \quad \delta^n_r(T)$$

will be used if $r_1 = \cdots = r_n = r$. As before, if $X_1 = \cdots = X_n = X$, then we simply replace $(X, \ldots, X)$ by X. And if Y is the scalar field $\mathbb{K}$, then we delete it from our notation. So

$$\mathcal{L}^n_p(X) \quad \text{and} \quad \mathcal{D}^n_p(X)$$

will denote the spaces of all n-linear forms on X^n (the n-fold Cartesian product of X with itself) which are $(p; p, \ldots, p)$-summing, respectively $(p, \ldots, p)$-dominated.

The same kind of simplifications apply to our next class of mappings (see [31] for a detailed exposition, and also [5]). Given $1 \leqslant p_1, \ldots, p_n \leqslant q < \infty$, we say that an n-linear mapping

$T: X_1 \times \cdots \times X_n \to Y$ is *multiple* $(q; p_1, \ldots, p_n)$*-summing* if there is a constant $K \geqslant 0$ such that for any $m_1, \ldots, m_n \in \mathbb{N}$ and $(x^j_{i_j})_{i_j=1}^{m_j} \subset X_j$, $1 \leqslant j \leqslant n$, we have

$$\left(\sum_{j=1}^{n}\sum_{i_j=1}^{m_j} \|T(x^1_{i_1}, \ldots, x^n_{i_n})\|^q\right)^{1/q} \leqslant K \cdot \prod_{j=1}^{n} \|(x^j_{i_j})_{i_j=1}^{m_j}\|^w_{p_j}.$$

We denote the smallest admissible K by $\pi_{(q;p_1,\ldots,p_n)}(T)$. Again, the set

$$\Pi^n_{(q;p_1,\ldots,p_n)}(X_1, \ldots, X_n; Y)$$

of all multiple $(q; p_1, \ldots, p_n)$-summing n-linear maps $X_1 \times \cdots \times X_n \to Y$ is a Banach space with $\pi_{(q;p_1,\ldots,p_n)}$ as a norm. We will mainly be concerned with the case $Y = \mathbb{K}$, $X_1 = \cdots = X_n = X$ and $q = p = p_1 = \cdots = p_n$, and then we write

$$\Pi^n_p(X)$$

for the above Banach space, or $[\Pi^n_p(X), \pi^n_p(\cdot)]$ if we wish to specify the norm.

It can be proved (see, for example, [27]) that, for Hilbert spaces $H_1, \ldots, H_n, H$, an n-linear mapping $T: H_1 \times \cdots \times H_n \to H$ is in $\Pi^n_2(H_1, \ldots, H_n; H)$ if and only if, regardless of how we select in each H_j an orthonormal basis $(e^j_{i_j})_{i_j \in I_j}$ in H_j, we have

$$\sum_{i_1 \in I_1, \ldots, i_n \in I_n} \|T(e^1_{i_1}, \ldots, e^n_{i_n})\|^2 < \infty.$$

Such T is called a (multilinear) *Hilbert–Schmidt operator*.

Of course, these concepts generalize corresponding ones which are well known from the theory of operator ideals. If $1 \leqslant p \leqslant \infty$, then $[\mathcal{L}^1_{(p;p)}(\cdot;\cdot), \|\cdot\|_{(p;p)}] = [\Pi^1_{(p;p)}(\cdot;\cdot), \pi_{p;p}]$ is one of the most important operator ideals: the Banach ideal

$$[\Pi_p, \pi_p]$$

of (absolutely) *p-summing operators*. $\mathcal{D}^1_p$ is just the ideal

$$\Pi_p$$

of *p-summing operators*; see [33, (17.4)], [17, Chapter 19], [19, Chapter 9].

Recall that $\Gamma_p(X, Y)$, $1 \leqslant p \leqslant \infty$, consists of those Banach space operators $u: X \to Y$ for which there is an $L_p(\mu)$-space Z and operators $v: Z \to Y^{**}$, $w: X \to Z$ such that $k_Y u = v \circ w$; here $k_Y: Y \hookrightarrow Y^{**}$ is the canonical evaluation map. These operators constitute a Banach ideal

$$[\Gamma_p, \gamma_p]$$

where, for $u \in \Gamma_p(X, Y)$, the norm $\gamma_p(u)$ is the infimum of all products $\|v\| \cdot \|w\|$, with v and w as above.

There is no need for passing to Y^{**} if $p = 2$ which, for our topic, is the most important case. Due to S. Kwapień is the result that $u \in \mathcal{L}(X, Y)$ belongs to $\mathcal{D}_2$ iff it admits a factorization $u: X \xrightarrow{w} Z \xrightarrow{v} Y$ where w and v's adjoint v^* belong to Π_2, and that in this case the $\mathcal{D}_2$-norm of u, which we denote by

$$\delta_2(u),$$

is the infimum (in fact, the minimum, see [17, p. 244]) of all products $\pi_2(v^*) \cdot \pi_2(w)$, v and w being admissible factors. Moreover, Z can be chosen to be a Hilbert space. As a consequence of Grothendieck's theorem, we also obtain that $u \in \mathcal{L}(X,Y)$ is in $\mathcal{D}_2$ iff a factorization $u : X \xrightarrow{w} Z \xrightarrow{v} Y$ is available where Z is a Banach space, v is in $\Gamma_1(Z,Y)$ and w is in $\Gamma_\infty(X,Z)$. Finally, $\mathcal{D}_2$ can also be characterized as the largest extension of Schatten trace class operators on Hilbert spaces to an ideal of Banach space operators; cf. [33, 17.5.2].

3. The bilinear extension property

The problem of characterizing BEP becomes accessible through a triviality: given Banach spaces X and Y, we can associate with each bilinear form $T \in \mathcal{L}^2(X,Y)$ the operator $u_T \in \mathcal{L}(X,Y^*)$ which is given by $\langle u_T(x), y\rangle := T(x,y)$ for $x \in X$ and $y \in Y$. The resulting map $\mathcal{L}^2(X,Y) \to \mathcal{L}(X,Y^*) : T \mapsto u_T$ is an isometric isomorphism.

We combine this with the preceding comments and recall that the space $L_\infty(\mu)$ associated with (e.g.) a finite measure μ is an injective Banach space in order obtain a first characterization of extendibility of bilinear forms (see also [13]):

Proposition 3.1. *$T \in \mathcal{L}^2(X,Y)$ is extendible iff $u_T \in \mathcal{D}_2(X,Y^*)$.*

Using the maximality of the ideal $\mathcal{D}_2$, together with the principle of local reflexivity, we therefore may state:

Corollary 3.2. *For any choice of Banach spaces X, Y, the following statements are equivalent:*

(i) *(X,Y) has BEP.*
(ii) *$\mathcal{L}(X,Y^*) = \mathcal{D}_2(X,Y^*)$.*
(iii) *For any $k,l = 0,1,\dots$, $(X^{(2k)}, Y^{(2l)})$ has BEP.*
(iv) *$(X^{(2k)}, Y^{(2l)})$ has BEP for some $k,l = 0,1,\dots$.*

Here $X^{(n)}$ denotes the nth dual of X.

Since the canonical map $k_X : X \hookrightarrow X^{**}$ is in $\mathcal{D}_2$ iff $\dim X < \infty$, we see that the pair (X,X^*) has BEP iff X is finite dimensional. It is open if an infinite dimensional Banach space X and its dual X^* can simultaneously have BEP; see also the remarks preceding 3.7.

An application of trace duality to 3.2.(ii) yields

Proposition 3.3. *If (X,Y) has BEP, then $\Gamma_2(Y^*,X) = \mathcal{I}_1(Y^*,X)$.*

Here $\mathcal{I}_1$ is the ideal of all Banach space operators $u : X \to Y$ which are 1-integral in the sense of Grothendieck: there exists a factorization of $k_Y u$ of the form $X \to L_\infty(\mu) \hookrightarrow L_1(\mu) \to Y^{**}$, where μ is a probability measure and "$\hookrightarrow$" represents the corresponding formal identity.

The converse of Proposition 3.3 fails in general, as we will see below, but it is true if X^* or Y^* has the metric approximation property; see [24].

The preceding result has interesting consequences. Recall that a Banach space X is said to *verify GT* (*'Grothendieck's Theorem'*) if $\mathcal{L}(X,\ell_2) = \Pi_1(X,\ell_2)$ holds; cf. [35]. In such a case, we shall also write $X \in GT$, or we say that X is a GT space, etc. Trace duality reveals that X is a GT space iff every operator from an $\mathcal{L}_\infty$ space into X is in $\mathcal{D}_2$ (equivalently, in Π_2); see, e.g., Proposition 20.18 in [17]. Classical examples of GT spaces are provided by $\mathcal{L}_1$ spaces (Grothendieck's theorem!), but there are more.

Theorem 3.4. *If (X, Y) has BEP, then X^* and Y^* verify GT.*

Proof. It suffices to look at infinite dimensional spaces. By Dvoretzky's theorem, X contains the ℓ_2^n's uniformly. Since the ideal Π_1 is the injective hull of $\mathcal{I}_1$ [33], it follows that every operator $Y^* \to \ell_2$ is 1-summing, that is, $Y^* \in GT$.

By symmetry, X^* verifies GT as well. □

We say that a Banach space X is a *HS space* if every operator $\ell_2 \to \ell_2$ which factorizes through X is a Hilbert–Schmidt operator [25]. Easily, this is equivalent to having $\mathcal{L}(X, \ell_2) = \Pi_2(X, \ell_2)$. Moreover, X is a *HS* space iff X^* has this property. Using the fact that $\pi_2(\mathrm{id}_{\ell_2^n}) = \sqrt{n}$, it is readily seen that no *HS* space can contain uniformly complemented copies of all ℓ_2^n's. By results of Pisier on K-convexity, every infinite dimensional *HS* space contains the ℓ_1^n's uniformly. Every *GT* space is a *HS* space, but not conversely. However, *HS* spaces of cotype 2 are *GT* spaces.

A variant of the preceding proof of 3.4 proceeds as follows: If (X, Y) has BEP, then, by Dvoretzky's theorem, X and Y are *HS* spaces and so (in the infinite dimensional case) contain uniform copies of all ℓ_1^n's. But by trace duality, $X^* \in GT$ iff $\mathcal{L}(X, \ell_1) = \mathcal{D}_2(X, \ell_1)$, and this coincides with $\Pi_2(X, \ell_1)$ since the adjoint of any operator $\ell_2 \to \ell_1$ is 2-summing. So, would X^* fail to be a *GT* space, we would be able to find integers $k_1 < k_2 < \cdots$ along with a uniformly bounded sequence of operators $T_n : X \to \ell_1^{k_n}$ such that $\lim \pi_2(T_n) = \infty$. Since Y^* is *HS*, there are embeddings $J_n : \ell_1^{k_n} \hookrightarrow Y^*$ such that $\sup_n \|J_n\| < \infty$. The compositions $J_n \circ T_n$ would be bounded in the operator norm, but not in the 2-summing norm (injectivity of Π_2), which contradicts $\mathcal{L}(X, Y^*) = \Pi_2(X, Y^*)$.

We write $X \in GT \wedge C_2$ if X is a cotype 2 space which verifies GT. Actually, it is still an open problem to know whether or not *GT* spaces always do have cotype 2. For spaces in $GT \wedge C_2$, there is a straightforward converse of 3.4. Say that a Banach space operator is *approximable* if it can be approximated, uniformly on compact sets, by finite rank operators. It follows from 3.2(ii) that if (X, Y) has BEP, then every operator $X \to Y^*$ is approximable.

Proposition 3.5. *If X^* and Y^* are in $GT \wedge C_2$, then (X, Y) has BEP iff every operator $X \to Y^*$ is approximable.*

Proof. All what is left to show is that if X^* and Y^* are in $GT \wedge C_2$ and if every operator in $\mathcal{L}(X, Y^*)$ is approximable, then (X, Y) has BEP. But since X^* and Y^* have cotype 2, every approximable operator $X \to Y^*$ factors through some Hilbert space [35, Theorem 4.1]. We are dealing with *HS* spaces, so that our hypothesis implies $\mathcal{L}(X, Y^*) = \mathcal{D}_2(X, Y^*)$. □

The conditions can be slightly relaxed: just require that X^* and Y^* are *GT* spaces of cotype 2 and that one of them embeds into a Banach space having cotype 2 and the approximation property; see again [35].

Another case, where a converse of 3.4 holds, occurs if a certain weak form of lattice structure is available. Recall that a Banach space is said to have the *property GL* (also known as $g\ell_2$) if every 1-summing operator from that space into ℓ_2 factors through an L_1-space. *GL* is a self-dual property, and it is shared by all Banach lattices. The terminology originates from the paper [23] by Y. Gordon and D.R. Lewis.

We shall write $X \in GT \wedge GL$ if the Banach space X verifies both, *GT* and *GL*. It is readily seen that this happens iff $\mathcal{L}(X, \ell_2) = \Gamma_1(X, \ell_2)$, and that such a space X is in $GT \wedge C_2$. Compare with [19, 17.11 and 17.12]. Recall that it is an open question if the only Banach spaces verifying GT and GL are $\mathcal{L}_1$-spaces (compare [35, p. 114]). A positive answer would render the equivalence of (ii) and (iii) in the next proposition trivial.

Proposition 3.6. *For every Banach space X the following are equivalent:*

(i) *X^* verifies GT.*
(ii) *(X, Y) has BEP, for every $\mathcal{L}_\infty$ space Y.*
(iii) *(X, Y) has BEP, for every Banach space such that $Y^* \in GT \wedge GL$.*

Proof. $X^* \in GT$ is equivalent to saying that every operator in $\mathcal{L}(\ell_2, X)$ has a 1-summing adjoint. Suppose that $Y^* \in GT \wedge GL$. Then $\mathcal{L}(Y^*, \ell_2) = \Gamma_1(Y^*, \ell_2)$, and this implies that if $v \in \mathcal{L}(X, Y^*)$, then $uvw \in \mathcal{I}_1(\ell_2, \ell_2)$ for all $u \in \mathcal{L}(Y^*, \ell_2)$ and $w \in \mathcal{L}(\ell_2, X)$. Consequently, v is in $\mathcal{D}_2(X, Y^*)$. This proves (i) $\Rightarrow$ (iii).

Since every $\mathcal{L}_1$ space verifies GT and GL, (iii) implies (ii). Finally, (ii) yields $\Gamma_1(X, \cdot) \subset \mathcal{D}_2(X, \cdot)$ which is (i), by duality. □

In general, the projective tensor norm $\tilde{\otimes}_\pi$ does not behave well with respect to the formation of subspaces. But it is immediate from the definition that (X, Y) has BEP iff $X \tilde{\otimes}_\pi Y$ is a subspace of $\widetilde{X} \tilde{\otimes}_\pi \widetilde{Y}$ whenever X is a subspace of $\widetilde{X}$ and Y is a subspace of $\widetilde{Y}$.

On the other hand, the injective tensor norm $\tilde{\otimes}_\varepsilon$ respects the formation of subspaces. Therefore, (X, Y) has BEP whenever X and Y are Banach spaces such that $X \tilde{\otimes}_\pi Y = X \tilde{\otimes}_\varepsilon Y$. In fact, in this case every operator $X \to Y^*$ is even 1-integral.

G. Pisier has shown that every Banach space of cotype 2 embeds into an infinite dimensional Banach space P such that $P \tilde{\otimes}_\pi P = P \tilde{\otimes}_\varepsilon P$ and both, P and P^* verify $GT \wedge C_2$, cf. [35]. Such a space will be referred to as a *Pisier space*.

The simple fact that (P, P^*) does not have BEP (since dim $P = \infty$) reveals that the converse implication fails in 3.3 as well as in 3.4. Note also that 3.6 yields the (known) result that no Pisier space can have GL. Moreover, by combination with our earlier observations we can now see that if X and Y are cotype 2 spaces, then (X, Y) has BEP iff $X \tilde{\otimes}_\pi Y = X \tilde{\otimes}_\varepsilon Y$. But this leads immediately to several questions which can be added to other open problems in this area: is it true that if P is a Pisier space, then P^* has BEP (i.e., is P^* a Pisier space)? Does (P, Q) have BEP when P and Q are Pisier spaces? More generally, is it possible that infinite dimensional Banach spaces X and Y exist such that (X, Y) and (X^*, Y^*) have BEP? By 3.3, we are thus asking for spaces X, Y such that $X \tilde{\otimes}_\varepsilon Y = X \tilde{\otimes}_\pi Y$ and $X^* \tilde{\otimes}_\varepsilon Y^* = X^* \tilde{\otimes}_\pi Y^*$. Our guess is that none of these questions has a positive answer.

Within our setting, the presence of $GT \wedge C_2$ has yet another consequence:

Proposition 3.7. *The following statements on a Banach space X are equivalent:*

(i) *(X, Y) has BEP for every Y such that $Y^* \in GT \wedge C_2$.*
(ii) *Every operator from X into any cotype 2 space is 2-summing.*

Proof. (i) $\Rightarrow$ (ii). Suppose that Z is a cotype 2 space. Take a Pisier space P with $Z \subset P$. Since $P^* \in GT \wedge C_2$, our assumption yields $\mathcal{L}(X, P^{**}) = \mathcal{D}_2(X, P^{**})$ which implies $\mathcal{L}(X, Z) = \Pi_2(X, Z)$.

(ii) $\Rightarrow$ (i). If Y^* verifies $GT \wedge C_2$, then every operator $X \to Y^*$ is in Π_2, and even in $\mathcal{D}_2$ because of $Y^* \in HS$. □

Again, even in this case, we do not know if X^* must have cotype 2.

A classical application of 3.7 occurs when X is a subspace of a space $C(K)$ such that $C(K)/X$ is reflexive (compare with [19, 15.13]): every operator from X into a cotype 2 space is 2-summing.

Moreover, by results of J. Bourgain [6,7], 3.7 also applies if we take X to be the space of all bounded analytic functions on the open unit disk D in $\mathbb{C}$, or the disk algebra on D, since again every operator from X into a cotype 2 space is 2-summing. See also [38, III.I.19]. It is known that in this case $X^* \in GT \wedge C_2$, and it was shown by A. Pelczyński [29] that X fails GL.

It might be possible that X^* is in $GT \wedge GL$ iff (X, Y) has BEP whenever Y^* verifies GT. This would imply that GT spaces without cotype 2 do exist, but we do not know how to get access. We can only prove a weaker result. Let $\mathcal{G}$ be the operator ideal which consists of all $v \in \mathcal{L}(X, Y)$ such that for every $u \in \mathcal{L}(Y, \ell_2)$ the composition $u \circ v$ is in $\Pi_1(X, \ell_2)$ (thus a Banach space verifies GT iff its identity is in $\mathcal{G}$). One can show that a Banach space X is in $GT \wedge GL$ if and only if every operator with domain X which is in $\mathcal{G}$ actually belongs to $\mathcal{D}_2$. We omit the details.

In view of topics to be discussed in the subsequent sections, we devote the rest of this section to a proof of

Theorem 3.8. *Suppose that X and Y are infinite dimensional Banach spaces. If (X, Y) has BEP, then $X \tilde{\otimes}_\pi Y$ contains uniformly complemented copies of the ℓ_2^n's.*

In particular, $X \tilde{\otimes}_\pi Y$ fails to be *HS* space!

It is easy to see that if X, Y, Z are Banach spaces such that (X, Y, Z) has TEP, then $(X, Y), (Y, Z), (Z, X)$ all have BEP (we shall provide a straightforward argument below). One might conjecture that $(X \tilde{\otimes}_\pi Y, Z)$, for example, enjoys BEP as well. The above theorem tells us that this is false for infinite dimensional spaces: $X \tilde{\otimes}_\pi Y$ cannot be *HS*. The latter can also be obtained using that if X and Y are infinite dimensional Banach spaces, then Dvoretzky's theorem asserts that the ℓ_1^n's are uniformly complemented in $X \tilde{\otimes}_\pi Y$.

In the proof of 3.8, we require the following lemma. We omit the details since the proof is essentially the same as the one of Lemma 1.1 in [8].

Lemma 3.9. *Let X, Y be Banach spaces such that every operator $T : X \to Y^*$ is 2-summing. Let (f_n) be a weakly 2-summable sequence in X and (g_n) be a bounded sequence in Y. Then $(f_n \otimes g_n)$ is weakly 2-summable in $X \tilde{\otimes}_\pi Y$.*

Proposition 3.10. *Let X, Y be Banach spaces such that*

(1) *every operator $T : X \to Y^*$ is 2-summing,*
(2) *Y contains uniformly the ℓ_1^n's.*

Then $X \tilde{\otimes}_\pi Y$ contains uniformly complemented copies of the ℓ_2^n's.

The assumptions are clearly satisfied if (X, Y) has BEP; therefore 3.8 is a corollary to 3.10.

Proof. The result, and its proof, refine some of the main results in [8].

Suppose that Y contains the ℓ_1^n's λ-uniformly, $\lambda > 1$. Then there exist $M > 0$ and for each $n \in \mathbb{N}$ and $N = N(n) > n$ in $\mathbb{N}$ together with a surjective operator $q_n : \ell_1^N \to \ell_2^n$ such that

$\|q_n\| \leqslant M$. Moreover, there are a constant C (independent of n) and, for each n, vectors $a_m \in \ell_1^N$, $1 \leqslant m \leqslant n$, such that

(1) $q_n(a_m) = e_m$ $(1 \leqslant m \leqslant n)$, where $(e_m)_{m=1}^n$ is the canonical ℓ_2^n-basis.
(2) $\sup_m \|a_m\| \leqslant C$.

Actually, if we are willing to increase $n \mapsto N(n)$, then we can choose C and M as close to one as we wish.

It follows that $\sup_n \pi_2(q_n) \leqslant K_G M$, where K_G is Grothendieck's constant. By the Π_2-extension theorem, each q_n is the restriction of a surjection $Q_n : Y \to \ell_2^n$ such that $\|Q_n\| \leqslant \lambda K_G M$ and such that there are $b_1, \dots, b_n \in Y$ satisfying

(1) $Q_n(b_m) = e_m$ $(1 \leqslant m \leqslant n)$.
(2) $\sup_m \|b_m\| \leqslant \lambda C$.

We may assume that X is infinite dimensional. By Dvoretzky's theorem, X contains for each n a subspace E_n which is 2-isomorphic to ℓ_2^n. Let $(f_m)_{m=1}^n$ be the basis in E_n obtained from the standard basis of ℓ_2^n via the corresponding isomorphism. We are going to work with Hahn–Banach extensions $f_m^* \in X^*$ of the associated biorthogonal functionals in E_n^*, $1 \leqslant m \leqslant n$.

The proof can now be completed as follows. Write $(\cdot|\cdot)$ for the scalar product of ℓ_2^n. Note that $e_m \mapsto f_m \otimes b_m$ gives rise to a unique linear map $\theta_n : \ell_2^n \to X \tilde{\otimes}_\pi Y$, and that linearization of the bilinear map $(x, y) \mapsto (f_m^*(x)(Q_n(y)|e_m))_{m=1}^n$ yields a continuous linear map $\varphi_n : X \tilde{\otimes}_\pi Y \to \ell_2^n$. From 3.9, it follows by standard reasonings that θ_n is continuous as well. Moreover, both mappings allow uniform estimates of their norms. Since $\varphi_n \circ \theta_n = \mathrm{Id}_{\ell_2^n}$, the identity in ℓ_2^n, we are done. $\square$

In passing, we notice that not just φ_n but even the above bilinear map $X \times Y \to \ell_2^n : (x, y) \mapsto (f_m^*(x)(Q_n(y)|e_m))_{m=1}^n$ is onto.

4. Extendibility and summability: The case $n \geqslant 3$

This section contains the main results of the paper, as they are summarized in Diagram 2.

Based on a multilinear version of Grothendieck's inequality as given in [3,11] and [36], the third-named author has proved the following result in [32]:

Theorem 4.1. *Let $\lambda_1, \dots, \lambda_n \geqslant 1$ be given and, for each $1 \leqslant j \leqslant n$, let X_j be an $\mathcal{L}_{\infty,\lambda_j}$ space. Then every multilinear form $T \in \mathcal{L}^n(X_1, \dots, X_n)$ is $(1; 2, \dots, 2)$-summing, and with $\lambda = \prod_{j=1}^n \lambda_j$, we have $\|T\|_{(1;2,\dots,2)} \leqslant \lambda K_G^{n-1} \|T\|$.*

Again, K_G is the Grothendieck constant.

In particular, if X is a Banach space and $\mathcal{L}_{\mathrm{ext}}^n(X)$ is the collection of all extendible n-linear forms on X, then

$$\mathcal{L}_{\mathrm{ext}}^n(X) \subseteq \mathcal{L}_{(1;2,\dots,2)}^n(X) \subseteq \mathcal{L}^n(X).$$

So all what is needed to complete our program is to justify the arrows appearing in Diagram 2. We start by an improvement of Theorem 4.1.

Theorem 4.2. *Suppose that X is an infinite dimensional Banach space with BEP. Then, for $n \geqslant 2$,*

$$\mathcal{L}^n(X) = \mathcal{L}^n_{(1;2,\dots,2)}(X).$$

Proof. We follow an induction argument from [36].

The case $n = 2$ is just the definition of BEP. Suppose then that the result is true for $n - 1$ ($n \geqslant 2$): there exists a constant $C > 0$ such that $\|T\|_{\mathcal{L}^{n-1}_{(1;2,\dots,2)}} \leqslant C\|T\|$ for every $T \in \mathcal{L}^{n-1}(X)$.

Take now any $T \in \mathcal{L}^n(X)$ with $\|T\| \leqslant 1$ and consider a finite collection of vectors $(x_i^j)_{i=1}^m$ in X, $1 \leqslant j \leqslant n$, such that $\|(x_i^j)_i\|_2^w \leqslant 1$ for each j. We claim that there exists a constant C', which is independent of T and of the chosen sequence, such that $\sum_{i=1}^m |T(x_i^1, \dots, x_i^n)| \leqslant C'$.

We define the operator $u : \ell_2^m \to X$ via $e_i \mapsto x_i^1$ for $1 \leqslant i \leqslant m$. It is standard that $\|u\| = \|(x_i^1)_i\|_2^w \leqslant 1$.

We also consider formal inclusion $i : \ell_1 \to \ell_2$ and the maps

$$v : X \to \mathcal{L}^{n-1}(X) \quad \text{given by } v(x) = T(x, \cdot, \dots, \cdot),$$
$$w : \mathcal{L}^{n-1}(X) \to \ell_1 \quad \text{given by } w(S) = \big(S(x_i^2, \dots, x_i^n)\big)_i.$$

Both maps are linear. Easily, v is continuous with norm no greater than 1. Also, using the induction hypothesis, we have that w verifies $\|w\| \leqslant C$.

X has BEP, and this implies by Theorem 3.4 that X^* is a *GT* space. This is equivalent to $\mathcal{L}(X, \ell_1) = \mathcal{D}_2(X, \ell_1)$, so that there is a constant γ such that $\delta_2(A) \leqslant \gamma\|A\|$ for all $A \in \mathcal{L}(X, \ell_1)$. Now $w \circ v$, being in $\mathcal{D}_2(X, \ell_1)$, admits a factorization $w \circ v = b \circ a$ with $a \in \Pi_2(X, H)$ and $b^* \in \Pi_2(Y^*, H)$ where H is a Hilbert space and $\pi_2(a)\pi_2(b^*) = \delta_2(w \circ v) \leqslant \gamma\|w \circ v\| \leqslant C'$ where $C' = \gamma C$.

We have thus established a factorization of $A := i \circ w \circ v \circ u : \ell_2^m \to \ell_2$:

$$\begin{tikzcd}
 & & H \arrow[drr, "b"] & & & \\
\ell_2^m \arrow[r, "u"] & X \arrow[ur, "a"] \arrow[r, "v"] & \mathcal{L}^{n-1}(X) \arrow[rr, "w"] & & \ell_1 \arrow[r, "i"] & \ell_2
\end{tikzcd}$$

Now, $a \circ u$ and $i \circ b$ are both of 2-summing operators between Hilbert spaces and thus have finite Hilbert–Schmidt norm. Therefore the trace class norm of A can be estimated as follows:

$$\sum_i^m \big|T(x_i^1, \dots, x_i^n)\big| = \big|\mathrm{tr}(A)\big| \leqslant \pi_2(a \circ u)\pi_2(i \circ b) \leqslant \pi_2(a)\pi_2(b^*) \leqslant C'.$$

This completes the proof. □

This theorem enables us to establish the remaining implications displayed in Diagram 2.

Theorem 4.3. *Let X be an infinite dimensional Banach space and $n \geqslant 3$. Then $\mathcal{L}^n_{\mathrm{ext}}(X) = \mathcal{L}^n_{(1;2,\dots,2)}(X)$ if and only if X has nEP.*

We require three lemmas.

Lemma 4.4. *If X is a HS space, then $\mathcal{L}^2(X) = \Pi_2^2(X)$.*

Proof. Fix $T \in \mathcal{L}^2(X)$ and $(x_i^j)_{i=1}^\infty \in \ell_2^w(X)$ for $j = 1, 2$. The linear maps $u_j : \ell_2 \to X$ given by $e_i \mapsto x_i^j$ $(j = 1, 2)$ are bounded with $\|u_j\| = \|(x_i^j)_i\|_2^w$, $j = 1, 2$. We have to check that $\sum_{i,k} |T(x_i^1, x_k^2)|^2 < \infty$. But

$$\sum_{i,k} \left|T\left(x_i^1, x_k^2\right)\right|^2 = \sum_{i,k} \left|T\left(u_1(e_i), u_2(e_k)\right)\right|^2 = \sum_{i,k} \left|\left[T \circ (u_1, u_2)\right](e_i, e_k)\right|^2,$$

and we know that the last expression is finite if and only if $T \circ (u_1 \times u_2)$ is Hilbert–Schmidt. This is equivalent to saying that the associated operator $u_{T\circ(u_1\times u_2)}$ is 2-summing. But this is the case, since $u_{T\circ(u_1\times u_2)}$ is just the composition $u_2^* \circ u_T \circ u_1$ and since X is a *HS* space. □

Lemma 4.5. *If a Banach space X fails BEP, then $\Pi_2^2(X) \setminus \mathcal{L}^2_{(1;2,2)}(X)$ is non-empty.*

Proof. By our assumptions and the definition of BEP we see that $\mathcal{L}^2_{(1;2,2)}(X)$ is a proper subset of $\mathcal{L}^2(X)$. Thus, if X is even a *HS* space, then our claim follows from 4.4.

Suppose next that X is not a *HS* space. Then X^* is not *HS*, either. Accordingly, there are integers $k_1 < k_2 < \cdots$ and operators $u_n : X^* \to \ell_2^{k_n}$ such that $M := \sup_n \|u_n\| < \infty$ but $(\pi_2(u_n))_n$ is unbounded. Dvoretzky's theorem provides us with isomorphic embeddings $j_n : \ell_2^{k_n} \hookrightarrow X$ such that $\sup_n \|j_n\| \leqslant 2$, say. The operators $s_n := j_n \circ u_n : X^* \to X$ satisfy $\sup_n \gamma_2(s_n) \leqslant 2M$, but $(\pi_2(s_n))_n$ is unbounded. Thus $\Pi_2(X^*, X)$ is properly contained in $\Gamma_2(X^*, X)$. An application of trace duality shows that $\mathcal{D}_2(X, X^*)$ is, therefore, a proper subset of $\Pi_2(X, X^*)$.

Let now $u : X \to X^*$ be 2-summing but not 2-dominated. Then the associated bilinear form $X \times X \to \mathbb{K}$ is not extendible (see 3.1), but the reasoning of the proof of 4.4 reveals that it is multiple 2-summing. □

Lemma 4.6. *Suppose we are given Banach spaces $X_1 \times \cdots \times X_n$, an n-linear form $T : X_1 \times \cdots \times X_n \to \mathbb{K}$ and a non-zero vector $y_0^* \in Y^*$. Then $y_0^*T : Y \times X_1 \times \cdots \times X_n \to \mathbb{K}$: $(y, x_1, \ldots, x_n) \mapsto \langle y_0^*, y\rangle \cdot T(x_1, \ldots, x_n)$ is an $(n+1)$-linear form. It is extendible if and only if T is extendible.*

Proof. It is clear that y_0^*T is $(n+1)$-linear. The least trivial of the remaining parts is to verify that T is extendible whenever y_0^*T is.

To this end, consider arbitrary superspaces $\widetilde{X}_1, \ldots, \widetilde{X}_n$ of $X_1, \ldots, X_n$, respectively, together with an $(n+1)$-linear extension $U : Y \times \widetilde{X}_1 \times \cdots \times \widetilde{X}_n \to \mathbb{K}$ of y_0^*T. Take any $y_0 \in Y$ such that $\langle y_0^*, y_0\rangle = 1$. Obviously, $\widetilde{X}_1 \times \cdots \times \widetilde{X}_n \to \mathbb{K} : (\tilde{x}_1, \ldots, \tilde{x}_n) \mapsto U(y_0, \tilde{x}_1, \ldots, \tilde{x}_n)$ is an n-linear form and extends T. □

Corollary 4.7. *A Banach space with nEP also has $(n-1)$EP.*

We are now ready for the

Proof of 4.3. For notational simplicity, we present the proof for $n = 3$ only.

If X has TEP, then an application of Theorem 4.1 gives $\mathcal{L}^3(X) = \mathcal{L}^3_{\mathrm{ext}}(X) = \mathcal{L}^3_{(1;2,2,2)}(X)$.

Suppose that conversely $\mathcal{L}^3_{\mathrm{ext}}(X) = \mathcal{L}^3_{(1;2,2,2)}(X)$ holds. The main step is to check that X has BEP. In fact, otherwise (by 4.5) some $S \in \Pi_2^2(X)$ will not belong to $\mathcal{L}^2_{(1;2,2)}(X)$ and so will fail to be extendible. Take a unit vector $x_0^* \in X^*$ and consider $T := x_0^*S \in \mathcal{L}^3(X)$. By 4.6, T is not extendible either. But it is $(1;2,2,2)$-summing because, if we are given $(x_i^j)_{i=1}^\infty \in \ell_2^w(X)$ for $j=1,2,3$, then we have, by Hölder's inequality,

$$\begin{aligned}
\sum_i |T(x_i^1,x_i^2,x_i^3)| &= \sum_i |x_0^*(x_i^1)S(x_i^2,x_i^3)| \\
&\leqslant \left(\sum_i |x_0^*(x_i^1)|^2\right)^{1/2} \cdot \left(\sum_i |S(x_i^2,x_i^3)|^2\right)^{1/2} \\
&\leqslant \|(x_i^1)_{i=1}^\infty\|_2^w \cdot \|S\|_{(2;2,2)} \cdot \|(x_i^2)_{i=1}^\infty\|_2^w \cdot \|(x_i^3)_{i=1}^\infty\|_2^w \\
&= \|S\|_{(2;2,2)} \cdot \|(x_i^1)_{i=1}^\infty\|_2^w \cdot \|(x_i^2)_{i=1}^\infty\|_2^w \cdot \|(x_i^3)_{i=1}^\infty\|_2^w.
\end{aligned}$$

Now, knowing that X has BEP, we get $\mathcal{L}^3(X) = \mathcal{L}^3_{(1;2,2,2)}(X)$ from 4.2. Thanks to our hypothesis, $\mathcal{L}^3_{\text{ext}}(X) = \mathcal{L}^3_{(1;2,2,2)}(X)$. Therefore X has TEP. □

Remark 4.8. We have shown that X has TEP iff $\mathcal{L}^3_{\text{ext}}(X) = \mathcal{L}^3_{(1;2,2,2)}(X)$. It is open if there is any Banach space which satisfies BEP but not TEP. We even do not know of any infinite dimensional Banach spaces, other than $\mathcal{L}_\infty$ spaces, which have TEP.

5. r-Dominated multilinear forms

In Diagram 1, the $(1;2,2)$-summing operators can be replaced by the 2-dominated ones since $\mathcal{L}^2_{(1;2,2)} = \mathcal{D}^2_2$. This prompts the question about the position taken by dominated operators inside of Diagram 2.

Let $n \geqslant 2$ be an integer. Using Khinchin's inequality (see, e.g., [19, Chapter 1]), and taking up some ideas of the bilinear case (which appears in [33]), we are going to prove:

Theorem 5.1. *Let real numbers $r_1,\dots,r_n \geqslant 1$ and Banach spaces $X_1,\dots,X_n$ be given. Then*

$$(*) \qquad \mathcal{D}^n_{(r_1,\dots,r_n)}(X_1,\dots,X_n) \subseteq \mathcal{L}^n_{(1;2,\dots,2)}(X_1,\dots,X_n)$$

and

$$\|T\|_{(1;2,\dots,2)} \leqslant B_r^n \cdot \delta_{(r_1,\dots,r_n)}(T)$$

for every $T \in \mathcal{D}^n_{(r_1,\dots,r_n)}(X_1,\dots,X_n)$. Here $r = \max\{r_1,\dots,r_n,n\}$ and B_r is the constant from Khinchin's inequality.

To prove this, we will need the following characterization of $(r_1,\dots,r_n)$-dominated n-linear maps from [22]:

Theorem 5.2. *Let $X_1,\dots,X_n,Y$ be Banach spaces, $r_1,\dots,r_n \geqslant 1$ be numbers and $T: X_1 \times \cdots \times X_n \to Y$ be an n-linear map. The following statements are equivalent:*

(a) *T is $(r_1,\dots,r_n)$-dominated.*

(b) *There exist a constant C and for each $1 \leqslant j \leqslant n$ a regular probability measure μ_j on $B_{X_j^*}$ such that for every $(x_1, \ldots, x_n) \in \prod_j X_j$,*

$$\|T(x_1, \ldots, x_n)\| \leqslant C \cdot \prod_{j=1}^{n} \left(\int_{B_{X_j^*}} |x_j^*(x_j)|^{r_j} \, d\mu_j(x_j^*) \right)^{1/r_j}.$$

(c) *There exist Banach spaces $Z_1, \ldots, Z_n$, a map $S \in \mathcal{L}^n(Z_1, \ldots, Z_n; Y)$ and for each $1 \leqslant j \leqslant n$ an operator $u_j \in \Pi_{r_j}(X_j, Z_j)$ such that $T = S \circ (u_1 \times \cdots \times u_n)$.*

Proof of 5.1. It is certainly enough to prove the result for $r_1 = \cdots = r_n = r \geqslant n \geqslant 2$.

We take $T \in \mathcal{D}_r^n(X_1, \ldots, X_n; \mathbb{K})$ and in each X_j an m-tuplet $(x_i^j)_{i=1}^m$ be given. We consider $D_m = \{-1, 1\}^m$ and the measure μ on D_m given by $\mu(e) = \frac{1}{2^m}$ for each $e = (e_1, \ldots, e_m) \in D_m$. If $\nu = \mu \otimes \cdots \otimes \mu$ is the $(n-1)$-fold product measure of copies of μ. As is shown in the proof of Theorem 3.10 in [1],

$$\begin{aligned}
&\sum_{i=1}^{m} T\big(x_i^1, \ldots, x_i^n\big) \\
&\quad = \int_{D_m^{n-1}} T\left(\sum_{i=1}^{m} e_i^1 x_i^1, \ldots, \sum_{i=1}^{m} e_i^{n-1} x_i^{n-1}, \sum_{i=1}^{m} e_i^1 \cdots e_i^{n-1} x_i^n \right) d\nu\big(e^1, \ldots, e^{n-1}\big).
\end{aligned}$$

Thus, if we set

$$x_{e^j} = \sum_{i=1}^{m} e_i^j x_i^j \quad \text{and} \quad a_{e^1,\ldots,e^{n-1}} = \sum_{i=1}^{m} e_i^1 \cdots e_i^{n-1} x_i^n,$$

we have that

$$\begin{aligned}
&\left| \sum_{i=1}^{m} T\big(x_i^1, \ldots, x_i^n\big) \right|^{r/2} \\
&\quad \leqslant \int_{D_m^{n-1}} \left| T\left(\sum_{i=1}^{m} e_i^1 x_i^1, \ldots, \sum_{i=1}^{m} e_i^{n-1} x_i^{n-1}, \sum_{i=1}^{m} e_i^1 \cdots e_i^{n-1} x_i^n \right) \right|^{r/2} d\nu\big(e^1, \ldots, e^{n-1}\big) \\
&\quad = \frac{1}{2^{m(n-1)}} \sum_{e^1,\ldots,e^{n-1} \in D_m} \big| T(x_{e^1}, \ldots, x_{e^{n-1}}, a_{e^1,\ldots,e^{n-1}}) \big|^{r/2}.
\end{aligned}$$

Now, since $T \in \mathcal{D}_r^n(X_1, \ldots, X_n; \mathbb{K})$, we can use the regular probability measures μ_j on $B_{X_j^*}$, $j \leqslant n$, which are provided by Theorem 5.2, to obtain

$$\begin{aligned}
&\sum_{e^1,\ldots,e^{n-1} \in D_m} \big| T(x_{e^1}, \ldots, x_{e^{n-1}}, a_{e^1,\ldots,e^{n-1}}) \big|^{r/2} \\
&\quad \leqslant \delta_r(T)^{r/2} \sum_{e^1,\ldots,e^{n-1} \in D_m} \left(\int_{B_{X_1^*}} \big| x_{e^1}\big(x_1^*\big) \big|^r \, d\mu_1\big(x_1^*\big) \right)^{1/2} \cdots
\end{aligned}$$

$$\cdot \left(\int_{B_{X_{n-1}^*}} \left| x_{e^{n-1}}\left(x_{n-1}^*\right)\right|^r d\mu_{n-1}\left(x_{n-1}^*\right)\right)^{1/2} \cdot \left(\int_{B_{X_n^*}} \left| a_{e^1,\dots,e^{n-1}}\left(x_n^*\right)\right|^r d\mu_n\left(x_n^*\right)\right)^{1/2}$$

$$\leqslant \delta_r(T)^{r/2} \cdot \left(\int_{B_{X_n^*}} \sum_{e^1,\dots,e^{n-1}\in D_m} \left| a_{e^1,\dots,e^{n-1}}\left(x_n^*\right)\right|^r d\mu_n\left(x_n^*\right)\right)^{1/2}$$

$$\cdot \prod_{j=1}^{n-1} \left(\int_{B_{X_j^*}} \sum_{e^j\in D_m} \left| x_{e^j}\left(x_j^*\right)\right|^r d\mu_j\left(x_j^*\right)\right)^{1/2}$$

$$\leqslant \left(\delta_r(T) \cdot \left\| (a_{e^1,\dots,e^{n-1}})_{e^1,\dots,e^{n-1}\in D_m} \right\|_r^w \cdot \prod_{j=1}^{n-1} \left\| (x_{e^j})_{e^j\in D_m}\right\|_r^w \right)^{r/2},$$

by repeated application of Hölder's inequality. Therefore,

$$\left| \sum_{i=1}^m T\left(x_i^1,\dots,x_i^n\right)\right|$$
$$\leqslant 2^{-\frac{2m(n-1)}{r}} \cdot \delta_r(T) \cdot \left\| (a_{e^1,\dots,e^{n-1}})_{e^1,\dots,e^{n-1}\in D_m} \right\|_r^w \cdot \prod_{j=1}^{n-1} \left\| (x_{e^j})_{e^j\in D_m}\right\|_r^w .$$

Now we apply Khinchin's inequality to obtain that, on the one hand, for each $j \in \{1,\dots,n-1\}$ and each $x_j^* \in X_j^*$, we have

$$\left(\frac{1}{2^m} \sum_{e^j\in D_m} \left| \langle x_j^*, x_{e^j}\rangle \right|^r \right)^{1/r} \leqslant B_r \left(\sum_{i=1}^m \left| \langle x_j^*, x_i^j\rangle\right|^2\right)^{1/2}$$

and thus

$$\left\| (x_{e^j})_{e^j\in D_m}\right\|_r^w \leqslant 2^{m/r} \cdot B_r \cdot \left\| \left(x_i^j\right)_{i=1}^m \right\|_2^w .$$

But on the other hand, this inequality also guarantees that, for each $x_n^* \in X_n^*$,

$$\left(\frac{1}{2^{m(n-1)}} \sum_{e^1,\dots,e^{n-1}\in D_m} \left| \langle x_n^*, a_{e^1,\dots,e^{n-1}}\rangle\right|^r \right)^{1/r} \leqslant B_r \cdot \left(\sum_{i=1}^m \left| \langle x_n^* x_i^n\rangle\right|^2\right)^{1/2}$$

thus

$$\left\| (a_{e^1,\dots,e^{n-1}})_{e^1,\dots,e^{n-1}\in D_m}\right\|_r^w \leqslant 2^{m(n-1)/r} \cdot B_r \cdot \left\| \left(x_i^n\right)_{i=1}^m \right\|_2^w .$$

We conclude that

$$\left| \sum_{i=1}^m T\left(x_i^1,\dots,x_i^n\right)\right| \leqslant B_r^n \cdot \delta_r(T) \cdot \prod_{j=1}^n \left\| \left(x_i^j\right)_{i=1}^m \right\|_2^w . \qquad \square$$

We finish by a supplement to the inclusion $(*)$.

Proposition 5.3. *Let X be an infinite dimensional Banach space. For any $n \geqslant 3$ and $r \in [1, \infty)$, $\mathcal{D}_r^n(X)$ is properly contained in $\mathcal{L}^n_{(1;2,\ldots,2)}(X)$.*

In the proof we use the following consequence of Dvoretzky's theorem:

Lemma 5.4. *For any infinite dimensional Banach space X, $m \geqslant 3$ and $1 \leqslant r < \infty$, there exists an m-linear form on X which is not r-dominated.*

Proof. We present the proof for $m = 3$; the generalization is straightforward.

Given $n \in \mathbb{N}$ choose $N = N(n) \in \mathbb{N}$ so that ℓ_2^n is 2-isomorphic (say) to a subspace of ℓ_∞^N and denote by i_n the corresponding embedding $\ell_2^n \hookrightarrow \ell_\infty^N$. Note that the $\pi_r(i_n) \to \infty$ when $n \to \infty$. By Dvoretzky's theorem, each ℓ_2^n is 2-isomorphic (say) to a subspace of X, and ℓ_2^N is 2-isomorphic to a subspace of X^*: let $j_n : \ell_2^n \hookrightarrow X$ and $k_n : \ell_2^N \hookrightarrow X^*$ the corresponding embeddings.

Identify ℓ_∞^N with the diagonal of $\ell_2^N \tilde{\otimes}_\varepsilon \ell_2^N$ and recall that $k_n \otimes k_n$ provides a 4-isomorphic embedding of $\ell_2^N \tilde{\otimes}_\varepsilon \ell_2^N$ into $X^* \tilde{\otimes}_\epsilon X^*$. In turn, the latter space embeds canonically into $(X \tilde{\otimes}_\pi X)^*$. Now, ℓ_∞^N is an injective Banach space, so that i_n extends to an operator $u_n : X \to \ell_\infty^N$, with nicely controllable norm. The operators $v_n := (k_n \otimes k_n) \circ u_n : X \to (X \tilde{\otimes}_\pi X)^*$ satisfy $\sup_n \|v_n\| < \infty$. But, by injectivity of the Banach ideal $[\Pi_r, \pi_r]$, $\sup_n \pi_r(v_n) = \infty$. This signifies that an operator $X \to (X \tilde{\otimes}_\pi X)^*$ exists which fails to be r-summing. By 5.2, the associated trilinear form cannot be r-dominated. □

We also require the next result which, as has been pointed out in [37], is an easy consequence of 5.2.

Corollary 5.5. *Let $1 \leqslant r < \infty$, $n \geqslant 2$ and $X_1, \ldots, X_n$ be Banach spaces. If an n-linear form $T : X_1 \times \cdots \times X_n \to \mathbb{K}$ is r-dominated, then the corresponding $(n-1)$-linear operator $\widehat{T} : X_1 \times \cdots \times X_{n-1} \to X_n^*$ defined as $\widehat{T}(x_1, \ldots, x_{n-1})(x_n) = T(x_1, \ldots, x_n)$ is also r-dominated.*

Proof of 5.3. For notational simplicity, we confine ourselves again to the case $n = 3$. First we consider the case where X does not have BEP. According to 4.5, we can find a bilinear form in $\Pi_2^2(X) \setminus \mathcal{L}^2_{(1;2,2)}(X)$. But $\mathcal{D}_r^2(X) \subset \mathcal{L}^2_{(1;2,2)}(X)$, so that we can find S in $\Pi_2^2(X) \setminus \mathcal{D}_r^2(X)$. As in the proof of 4.3, we fix a unit vector $x_0^* \in X^*$ and define $T \in \mathcal{L}^3(X)$ to be $T := x_0^* S$. We have already seen that it belongs to $\mathcal{L}^3_{(1;2,2,2)}(X)$. Using 5.5, it is easy to check that it is not r-dominated.

If X has BEP, then $\mathcal{L}^3(X) = \mathcal{L}^3_{(1;2,2,2)}(X)$ by 4.2. But we have seen in 5.4 that $\mathcal{D}_r^3(X)$ is properly contained in $\mathcal{L}^3(X)$. We are done. □

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